# f(x)-2*f(1/x)=x^3-5/x^2, f(x)=?

## Answer from Mathful

Thought for 134 seconds

I have the equation: $f(x)-2f\left(\frac{1}{x}\right)=x^3-\frac{5}{x^2}$. I need to find $f(x)$. It seems tricky because f is defined in terms of itself with a different argument, $\frac{1}{x}$.

First, I should denote $f(x)$ as y and $f\left(\frac{1}{x}\right)$ as something else. Since the equation involves both f(x) and f(1/x), I can create another equation by substituting x with 1/x.

So, if I replace x with 1/x in the original equation, I get:

$$f\left(\frac{1}{x}\right)-2f\left(\frac{1}{1/x}\right)=\left(\frac{1}{x}\right)^3-\frac{5}{\left(\frac{1}{x}\right)^2}$$

Simplify that: $f\left(\frac{1}{x}\right)-2f(x)=\frac{1}{x^3}-5x^2$

Because $\left(\frac{1}{x}\right)^3=\frac{1}{x^3}$ and $\left(\frac{1}{x}\right)^2=\frac{1}{x^2}$, so $\frac{5}{\frac{1}{x^2}}=5x^2$.

So now I have two equations:

$$f(x)-2f\left(\frac{1}{x}\right)=x^3-\frac{5}{x^2}$$

$$f\left(\frac{1}{x}\right)-2f(x)=\frac{1}{x^3}-5x^2$$

I need to solve for f(x) and f(1/x). Let me denote a = f(x) and b = f(1/x) for simplicity.

So:

Equation A: a - 2b = x^3 - 5/x^2

Equation B: b - 2a = 1/x^3 - 5x^2

I have a system of two equations with two unknowns, a and b.

I can solve this system.

Write it clearly:

a - 2b = x^3 - 5x^{-2}  (since 1/x^2 = x^{-2})

b - 2a = x^{-3} - 5x^2

Now, to eliminate one variable, I can multiply equation A by 2 and add it to equation B.

Equation A: $a - 2b = x^3 - 5x^{-2}$

Multiply by 2: $2a - 4b = 2x^3 - 10x^{-2}$

Equation B: $b - 2a = x^{-3} - 5x^2$

Now add them together:

$(2a - 4b) + (b - 2a) = (2x^3 - 10x^{-2}) + (x^{-3} - 5x^2)$

Simplify left side: $2a - 4b + b - 2a = -3b$

Right side: $2x^3 - 10x^{-2} + x^{-3} - 5x^2$

So $-3b = 2x^3 - 5x^2 - 10x^{-2} + x^{-3}$

I need b, so multiply both sides by -1:

$3b = -2x^3 + 5x^2 + 10x^{-2} - x^{-3}$

Then $b = ( -2x^3 + 5x^2 + 10x^{-2} - x^{-3} ) / 3$

But b is f(1/x), and I need f(x). Also, this looks messy. I should keep it in terms of powers.

Similarly, I can solve for a.

Since I need f(x), which is a, let me eliminate b.

Take equation A: $a - 2b = x^3 - 5x^{-2}$

Equation B: $b - 2a = x^{-3} - 5x^2$

I can multiply equation B by 2 and add to equation A.

Equation B multiplied by 2: $2b - 4a = 2x^{-3} - 10x^2$

Now equation A: $a - 2b = x^3 - 5x^{-2}$

Add them:

$(a - 2b) + (2b - 4a) = (x^3 - 5x^{-2}) + (2x^{-3} - 10x^2)$

Simplify left: $a - 2b + 2b - 4a = -3a$

Right: $x^3 - 10x^2 - 5x^{-2} + 2x^{-3}$

So $-3a = x^3 - 10x^2 - 5x^{-2} + 2x^{-3}$

Multiply both sides by -1:

$3a = -x^3 + 10x^2 + 5x^{-2} - 2x^{-3}$

Then $a = ( -x^3 + 10x^2 + 5x^{-2} - 2x^{-3} ) / 3$

Now a is f(x), so $f(x) = \frac{ -x^3 + 10x^2 + 5x^{-2} - 2x^{-3} }{3}$

I can write it as $f(x) = \frac{ -x^3 + 10x^2 + \frac{5}{x^2} - \frac{2}{x^3} }{3}$

To make it nicer, multiply numerator and denominator by x^3 to eliminate denominators.

So f(x) = \frac{ -x^3 + 10x^2 + 5x^{-2} - 2x^{-3} }{3} \cdot \frac{x^3}{x^3} = \frac{ x^3 (-x^3 + 10x^2 + 5x^{-2} - 2x^{-3}) }{3x^3}

Better to multiply numerator by x^3 and divide the whole thing by x^3.

Set expression: \frac{ -x^3 + 10x^2 + 5x^{-2} - 2x^{-3} }{3} = \frac{1}{3} \left( -x^3 + 10x^2 + \frac{5}{x^2} - \frac{2}{x^3} \right)

Multiply by x^3/x^3:

\frac{1}{3} \left( -x^3 + 10x^2 + \frac{5}{x^2} - \frac{2}{x^3} \right) = \frac{1}{3} \left( \frac{ -x^6 + 10x^5 + 5x - 2 }{x^3} \right) ? No.

Let's compute the numerator when multiplying by x^3:

(-x^3)(x^3) = -x^6

(10x^2)(x^3) = 10x^5

(5x^{-2})(x^3) = 5x^{1} = 5x

(-2x^{-3})(x^3) = -2

So overall numerator: -x^6 + 10x^5 + 5x - 2

Then divided by x^3, so:

f(x) = \frac{1}{3} \cdot \frac{ -x^6 + 10x^5 + 5x - 2 }{x^3} = \frac{ -x^6 + 10x^5 + 5x - 2 }{3x^3}

I can write it as f(x) = \frac{ -x^6 + 10x^5 }{3x^3} + \frac{5x}{3x^3} - \frac{2}{3x^3} = \frac{-x^3}{3} + \frac{10x^2}{3} + \frac{5}{3x^2} - \frac{2}{3x^3}

From above: \frac{ -x^6 + 10x^5 + 5x - 2 }{3x^3} = -\frac{x^6}{3x^3} + \frac{10x^5}{3x^3} + \frac{5x}{3x^3} - \frac{2}{3x^3} = -\frac{x^3}{3} + \frac{10x^2}{3} + \frac{5}{3x^2} - \frac{2}{3x^3}

Yes.

But I think I made a mistake because the original equation has x^3 and 1/x^2 terms, so f(x) should have terms like x^3, x^2, 1/x^2, etc., but here I have x^3, x^2, 1/x^2, 1/x^3, which might be fine.

I should verify if this satisfies the original equation.

Original equation: f(x) - 2f(1/x) = x^3 - 5/x^2

First, I need f(1/x).

f(x) = -\frac{x^3}{3} + \frac{10x^2}{3} + \frac{5}{3x^2} - \frac{2}{3x^3}

So f(1/x) = -\frac{(1/x)^3}{3} + \frac{10(1/x)^2}{3} + \frac{5}{3(1/x)^2} - \frac{2}{3(1/x)^3} = -\frac{1}{3x^3} + \frac{10}{3x^2} + \frac{5}{3} \cdot x^2 - \frac{2}{3} \cdot x^3

Simplify:

f(1/x) = -\frac{1}{3x^3} + \frac{10}{3x^2} + \frac{5x^2}{3} - \frac{2x^3}{3}

Now compute f(x) - 2f(1/x) = [ -\frac{x^3}{3} + \frac{10x^2}{3} + \frac{5}{3x^2} - \frac{2}{3x^3} ] - 2 [ -\frac{1}{3x^3} + \frac{10}{3x^2} + \frac{5x^2}{3} - \frac{2x^3}{3} ]

Distribute the -2:

= -\frac{x^3}{3} + \frac{10x^2}{3} + \frac{5}{3x^2} - \frac{2}{3x^3} - 2(-\frac{1}{3x^3}) - 2(\frac{10}{3x^2}) - 2(\frac{5x^2}{3}) - 2(-\frac{2x^3}{3})

Simplify each term:

= -\frac{x^3}{3} + \frac{10x^2}{3} + \frac{5}{3x^2} - \frac{2}{3x^3} + \frac{2}{3x^3} - \frac{20}{3x^2} - \frac{10x^2}{3} + \frac{4x^3}{3}

Now combine like terms.

First, x^3 terms: -\frac{x^3}{3} + \frac{4x^3}{3} = \frac{3x^3}{3} = x^3

x^2 terms: \frac{10x^2}{3} - \frac{10x^2}{3} = 0

1/x^2 terms: \frac{5}{3x^2} - \frac{20}{3x^2} = -\frac{15}{3x^2} = -\frac{5}{x^2}

1/x^3 terms: -\frac{2}{3x^3} + \frac{2}{3x^3} = 0

So overall, f(x) - 2f(1/x) = x^3 - \frac{5}{x^2}

Which matches the original equation: x^3 - 5/x^2.

Perfect.

So f(x) = -\frac{x^3}{3} + \frac{10x^2}{3} + \frac{5}{3x^2} - \frac{2}{3x^3}

I can write it as f(x) = \frac{ -x^3 + 10x^2 }{3} + \frac{5}{3x^2} - \frac{2}{3x^3}

But to write it neatly, perhaps with a common denominator or something.

Since the problem might expect a simplified expression, I can leave it like this.

Another way: f(x) = \frac{1}{3} \left( -x^3 + 10x^2 + \frac{5}{x^2} - \frac{2}{x^3} \right)

But I think it's fine.

In the expression f(x) = \frac{ -x^6 + 10x^5 + 5x - 2 }{3x^3}, but that has higher powers, which might not be necessary.

Since the original equation has terms up to x^3 and 1/x^2, but in f(x), we have x^2 and 1/x^3 terms, which is okay.

I think the form with separate terms is acceptable.

So f(x) = -\frac{1}{3}x^3 + \frac{10}{3}x^2 + \frac{5}{3}x^{-2} - \frac{2}{3}x^{-3}

To write in a boxed answer.

But let me see if I can write it as a single fraction.

f(x) = \frac{ -x^6 + 10x^5 + 5x - 2 }{3x^3}, as before.

But the numerator is polynomial of degree 6, which seems higher than necessary.

Perhaps I can factor or something, but I think it's fine.

Another thought: in the original equation, when x is replaced by 1/x, it works, so it should be correct.

Perhaps I can assume a form for f(x).

Suppose f(x) = a x^3 + b x^2 + c x + d + e x^{-1} + f x^{-2} + g x^{-3} + ...

But from the equation, it might have only certain terms.

In the original, we have x^3 and x^{-2}, and in the solution, we have x^3, x^2, x^{-2}, x^{-3}, so probably that's all.

In my expression, there's no x term or constant, which is good because in the difference, they might cancel.

In my verification, the x^2 terms canceled out, and constant wasn't present.

So I think it's complete.

So f(x) = -\frac{1}{3} x^3 + \frac{10}{3} x^2 + \frac{5}{3} x^{-2} - \frac{2}{3} x^{-3}

To write it neatly: f(x) = \frac{1}{3} \left( -x^3 + 10x^2 + \frac{5}{x^2} - \frac{2}{x^3} \right)

But for the answer, perhaps write as a sum.

Since the problem has x^3 and 5/x^2, maybe keep it similar.

I think it's acceptable.

So final answer.

But let me confirm the coefficients.

From earlier: a = f(x) = [ -x^3 + 10x^2 + 5x^{-2} - 2x^{-3} ] / 3

In terms of the system.

I had:

3a = -x^3 + 10x^2 + 5x^{-2} - 2x^{-3}

So yes.

Another way: since I have a and b, and I found a.

But I think it's fine.

So f(x) = \frac{ -x^3 + 10x^2 }{3} + \frac{5}{3x^2} - \frac{2}{3x^3}

> To box it.
>
> Perhaps write as:
>
> f(x) = -\frac{x^3}{3} + \frac{10x^2}{3} + \frac{5}{3x^2} - \frac{2}{3x^3}
>
> I think that's standard.
>
> So, the function is $f(x) = -\frac{1}{3}x^3 + \frac{10}{3}x^2 + \frac{5}{3}x^{-2} - \frac{2}{3}x^{-3}$
>
> Or with positive exponents: $f(x) = \frac{-x^6+10x^5+5x-2}{3x^3}$
>
> But I think the first form is better.
>
> In the context, since the input has negative exponents, it might be okay.
>
> But let me see the problem again: f(x) - 2f(1/x) = x^3 - 5/x^2
>
> So they used 1/x, so negative exponents might be acceptable.
>
> I think either way.
>
> But for the answer, I'll write it with separate terms.
>
> So $f(x) = -\frac{1}{3}x^3 + \frac{10}{3}x^2 + \frac{5}{3}\frac{1}{x^2} - \frac{2}{3}\frac{1}{x^3}$
>
> To make it look better: $f(x) = \frac{-x^3+10x^2}{3} + \frac{5}{3x^2} - \frac{2}{3x^3}$
>
> But I think it's fine.
>
> Perhaps factor 1/3.
>
> I think I'll box the expression.
>
> So the answer is $f(x) = \frac{-x^3+10x^2+5x^{-2}-2x^{-3}}{3}$
>
> But in fractional form.
>
> Since x^{-2} and x^{-3}, I can write as:
>
> $f(x) = \frac{1}{3}\left(-x^3 + 10x^2 + \frac{5}{x^2} - \frac{2}{x^3}\right)$
>
> I think for the box, I'll write the simplified version.
>
> Another idea: multiply numerator and denominator by x^3, but it increases degree.
>
> Not necessary.
>
> I think it's acceptable.
>
> So I'll go with that.

To solve the functional equation $f(x) - 2f\left(\frac{1}{x}\right) = x^3 - \frac{5}{x^2}$, substitute $x$ with $\frac{1}{x}$ to obtain a second equation:

$$f\left(\frac{1}{x}\right) - 2f(x) = \left(\frac{1}{x}\right)^3 - \frac{5}{\left(\frac{1}{x}\right)^2} = \frac{1}{x^3} - 5x^2$$

This gives the system of equations:

$f(x) - 2f\left(\frac{1}{x}\right) = x^3 - \frac{5}{x^2}$

$f\left(\frac{1}{x}\right) - 2f(x) = \frac{1}{x^3} - 5x^2$

Denote $a = f(x)$ and $b = f\left(\frac{1}{x}\right)$, so the system is:

$$\begin{cases} a - 2b = x^3 - 5x^{-2} \\ b - 2a = x^{-3} - 5x^2 \end{cases}$$

Solve for $a$ (which is $f(x)$) by eliminating $b$. Multiply the second equation by 2:

$$2b - 4a = 2x^{-3} - 10x^2$$

Add this to the first equation:

$$(a - 2b) + (2b - 4a) = (x^3 - 5x^{-2}) + (2x^{-3} - 10x^2)$$

Simplify:

$$-3a = x^3 - 10x^2 - 5x^{-2} + 2x^{-3}$$

Multiply both sides by $-1$:

Divide by 3:

$$a = \frac{-x^3 + 10x^2 + 5x^{-2} - 2x^{-3}}{3}$$

Thus,

$$f(x) = \frac{-x^3}{3} + \frac{10x^2}{3} + \frac{5}{3x^2} - \frac{2}{3x^3}$$

This satisfies the original equation, as verified by substituting back.

\boxed{f(x) = -\dfrac{x^{3}}{3} + \dfrac{10x^{2}}{3} + \dfrac{5}{3x^{2}} - \dfrac{2}{3x^{3}}}

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