x and y are two-digit natural numbers less than 50. i is the imaginary number in complex domain so that i^2 = -1. How many different pairs of (x, y) are there to meet the equation i^x + i^y = 0 ? A)50, B)100, C)200, D)400, E)500

Answer from Mathful

Thought for 320 seconds

The problem states that x and y are two-digit natural numbers less than 50. Also, i is the imaginary unit where i squared is -1. I need to find how many different pairs (x, y) satisfy the equation i^x + i_y = 0. The options are 50, 100, 200, 400, or 500.

First, recall the powers of i. Since i^4 = 1, the powers repeat every 4 cycles:

i^1 = i

i^2 = -1

i^3 = -i

i^4 = 1

i^5 = i, and so on.

The equation is i^x + i^y = 0, which means i^x = -i^y.

Since the powers are periodic with period 4, I should consider the exponents modulo 4.

Set a = x mod 4 and b = y mod 4. Then i^x depends only on a, and i^y on b.

Specifically:

If a ≡ 0 mod 4, i^x = i^4k = (i^4)^k = 1^k = 1

If a ≡ 1 mod 4, i^x = i

If a ≡ 2 mod 4, i^x = -1

If a ≡ 3 mod 4, i^x = -i

Similarly for y.

The equation i^x + i^y = 0 implies that i^x and i^y are negatives of each other.

So, possible pairs (i^x, i^y) that sum to zero are:

i^x = 1, i^y = -1

i^x = -1, i^y = 1

i^x = i, i^y = -i

i^x = -i, i^y = i

From the values above:

If i^x = 1, then a ≡ 0

i^x = i, a ≡ 1

i^x = -1, a ≡ 2

i^x = -i, a ≡ 3

Similarly for b.

So, the pairs:

i^x = 1, i^y = -1 ⇒ a ≡ 0, b ≡ 2

i^x = -1, i^y = 1 ⇒ a ≡ 2, b ≡ 0

i^x = i, i^y = -i ⇒ a ≡ 1, b ≡ 3

i^x = -i, i^y = i ⇒ a ≡ 3, b ≡ 1

So, essentially, for each pair (x,y), the residues modulo 4 must satisfy one of these conditions:

a=0, b=2

a=2, b=0

a=1, b=3

a=3, b=1

In other words, a and b must be different and not both even or both odd? Let's see:

Cases: (0,2), (2,0), (1,3), (3,1)

Notice that in all cases, a and b are such that a - b ≡ 2 mod 4 or b - a ≡ 2 mod 4? Let's compute the differences.

For (0,2): 0-2 ≡ -2 ≡ 2 mod 4

For (2,0): 2-0 ≡ 2 mod 4

For (1,3): 1-3 ≡ -2 ≡ 2 mod 4

For (3,1): 3-1 ≡ 2 mod 4

All differences are congruent to 2 modulo 4. But that's |a-b|? Actually, in each case, |a-b| = 2, but modulo 4, since a and b are between 0 and 3, |a-b| is either 2 or more, but actually in all these pairs, the difference is exactly 2 modulo 4? 0 and 2: difference 2, 1 and 3: difference 2, yes.

But is that sufficient? Suppose a and b differ by 2 mod 4, then:

If a≡0, b≡2: yes

a≡2, b≡0: yes

a≡1, b≡3: yes

a≡3, b≡1: yes

Now, are there other pairs where difference is 2? Like a≡0, b≡2; but b≡ -2 ≡ 2 mod 4, same.

What if a≡0, b≡ -2 ≡ 2 mod 4, same as above.

Is there a pair like a≡0, b≡2; a≡2, b≡0; etc.

But what about a≡0, b≡0? difference 0, not 2.

a≡0, b≡1: diff 1, not 2.

a≡0, b≡3: diff 3, not 2.

a≡1, b≡1: diff 0.

a≡1, b≡2: diff 1.

a≡1, b≡3: diff 2, which is included.

a≡2, b≡2: diff 0.

a≡2, b≡3: diff 1.

a≡3, b≡3: diff 0.

a≡3, b≡0: diff 3? |3-0|=3, but modulo 4, 3 - 0 = 3 ≡ 3 mod 4, not 2.

Earlier I computed differences, but in the cases where it works, difference is exactly 2, not modulo.

In modulo 4, when we say difference ≡ 2 mod 4, it could be ±2.

a - b ≡ 2 mod 4 or a - b ≡ -2 ≡ 2 mod 4? -2 ≡ 2 mod 4? Yes, since 4 divides 4, -2 + 4 = 2.

-2 mod 4 is 2, because -2 + 4 = 2.

So a - b ≡ 2 mod 4.

Is that sufficient?

If a - b ≡ 2 mod 4, then:

Possible: a=0,b=2: 0-2=-2≡2 mod 4? -2 + 4 = 2, yes.

$a=1,b=3:1-3=-2\equiv 2 \bmod 4$.

$a=2,b=0:2-0=2\equiv 2$.

$a=3,b=1:3-1=2\equiv 2$.

$a=2,b=2:2-2=0\neq 2$.

$a=0,b=0:0\neq 2$.

$a=0,b=1: -1\equiv 3\neq 2$.

$a=0,b=3: -3\equiv 1\neq 2$.

$a=1,b=0:1\equiv 1\neq 2$.

$a=1,b=1:0\neq 2$.

$a=1,b=2:-1\equiv 3\neq 2$.

$a=2,b=1:1\equiv 1\neq 2$.

$a=2,b=3:-1\equiv 3\neq 2$.

$a=3,b=0:3\equiv 3\neq 2$.

$a=3,b=2:1\equiv 1\neq 2$.

$a=3,b=3:0\neq 2$.

So only when $|a-b|=2$, which corresponds exactly to the pairs we have.

$a-b\equiv 2 \bmod 4$ implies $|a-b|=2$, since a,b in {0,1,2,3}, and difference is 2 or -2, but absolute value 2.

So yes, necessary and sufficient that $|a - b| = 2$, but since modulo 4, it's equivalent to $a - b \equiv 2 \bmod 4$.

In this set, $|a-b|=2$ is sufficient, as we have only those cases.

But in our case, we have exactly those four pairs.

Now, for each such pair (a,b) in residues, but a and b are determined by x and y modulo 4.

Now x and y are two-digit natural numbers less than 50. Two-digit natural numbers start from 10 to 49 inclusive, since less than 50, so 10 through 49.

How many such numbers? From 10 to 49 inclusive: $49 - 10 + 1 = 40$ numbers.

Confirm: 10,11,...,49. Number is $49-10+1=40$.

Yes.

Each of these numbers has a residue modulo 4, which is 0,1,2, or 3.

Now, I need to find for how many pairs (x,y) such that $|a - b| = 2$, where $a = x \bmod 4$, $b = y \bmod 4$.

Since the equation depends only on the residues, and we have the conditions.

To count the number of pairs where the residues satisfy one of the four cases: (a,b) = (0,2), (2,0), (1,3), (3,1).

Since the residues are independent, I can find how many x have each residue, and same for y, and then pair them accordingly.

Since x and y are chosen from the same set, and pairs are ordered, I assume, since (x,y) is ordered pair, probably, because the equation is not symmetric, but in this case, looking at the conditions, (0,2) and (2,0) are different, so likely ordered pairs.

In the pair (x,y), and x and y are distinct variables, so probably ordered pairs.

Moreover, in the answer choices, they are large, so likely ordered.

Now, let me denote the set S = {10,11,12,...,49}, size 40.

I need the number of elements in S with each residue modulo 4.

Let me compute.

First, find how many numbers from 10 to 49 inclusive are divisible by 4, i.e., residue 0.

The smallest number ≥10 divisible by 4: 12 (since 12/4=3), largest ≤49: 48 (48/4=12).

So, sequence: 12,16,...,48.

This is arithmetic with difference 4.

First term 12, last 48, common difference 4.

Number of terms: (48 - 12)/4 + 1 = 36/4 +1 = 9+1=10.

(48 - 12)/4 = 36/4=9, then +1=10, yes.

12,16,20,24,28,32,36,40,44,48. That's 10 numbers.

Now, residue 1: numbers ≡1 mod 4.

Smallest: 13 (13-1=12 divisible by 4), largest: 49 (49-1=48 divisible by 4).

13,17,...,49.

First 13, last 49, diff 4.

Number: (49 - 13)/4 +1 = 36/4 +1=9+1=10.

13,17,21,25,29,33,37,41,45,49. Yes, 10.

Residue 2: ≡2 mod 4.

Smallest: 10? 10 div 4 is 2*4=8, remainder 2, yes. Largest: 46? 46-2=44, 44/4=11, yes. 46≡2.

10,14,18,...,46.

First 10, last 46, diff 4.

(46-10)/4 +1 = 36/4 +1=9+1=10.

10,14,18,22,26,30,34,38,42,46. Yes, 10.

Residue 3: ≡3 mod 4.

Smallest: 11? 11-3=8, divisible by 4. Largest: 47? 47-3=44, yes.

11,15,19,...,47.

First 11, last 47, diff 4.

(47-11)/4 +1=36/4 +1=9+1=10.

11,15,19,23,27,31,35,39,43,47. Yes, 10.

So, exactly 10 numbers for each residue modulo 4. That's convenient.

Total 40 numbers, 10 in each residue class.

Now, for the pairs (x,y), we need:

Case 1: a=0, b=2 → number of x with res 0: 10, y with res 2: 10, so 10*10=100 pairs.

Case 2: a=2, b=0 → x res 2:10, y res 0:10, so 10*10=100 pairs.

Case 3: a=1, b=3 → x res 1:10, y res 3:10, so 100 pairs.

Case 4: a=3, b=1 → x res 3:10, y res 1:10, so 100 pairs.

Now, total pairs: 100 * 4 = 400.

And that's option D.

But is that all? The pairs are disjoint, since each has different residue combinations.

For example, if x has res 0 and y res 2, that's different from others.

No overlap.

So total 400 pairs.

But let me verify with an example.

Take x=10, which is ≡2 mod 4, since 10/4=2*4=8, remainder 2, so i^x = i^{10} = (i^4)^2 * i^2 = 1^2 * (-1) = -1.

Now, i^y must be - (-1) = 1? No, equation is i^x + i^y =0, so -1 + i^y =0, so i^y=1.

So y must be ≡0 mod 4.

For example, y=12, i^{12}=(i^4)^3=1^3=1, so -1 + 1=0, yes.

In this case, for x≡2, y≡0.

Similarly, if x≡0, y≡2, same thing.

Now, for x≡1, say x=13, i^{13}=i^{12}*i=1*i=i.

Then i^y must be -i, so y≡3, say y=11, i^{11}=i^{8}*i^{3}=1*(-i)=-i, so i + (-i)=0, yes.

Similarly for others.

Now, are there any restrictions? x and y are both from the set, and we have counted all such pairs.

Total possible pairs without restriction: 40 ∗ 40 = 1600.

Now, the residue pairs that work are only these four combinations out of possible 4×4=16 residue pairs.

Total residue combinations: each pair (a,b) has 4 choices for a, 4 for b, total 16 possibilities.

Out of which, how many satisfy the condition? We have four: (0,2),(2,0),(1,3),(3,1).

The others do not satisfy, as we saw earlier.

And since each residue class has exactly 10 elements, so for each fixed (a,b) pair, there are 10 choices for x, 10 for y.

So total satisfying: 4 * (10∗10) = 400.

Since there are 16 possible residue pairs, and 4 satisfy, and since each residue pair is equally likely (since each residue has same number), so fraction is 4/16=1/4 of all pairs.

Total pairs: 40 ∗ 40 = 1600.

1600 / 4 = 400, same.

Now, is there any pair where x=y? But if x=y, then i^x + i^x = 2i^x, which is zero only if i^x=0, impossible. But in our pairs, for example, (0,2) and (2,0), if x and y same, but residues different, so x≠y in all cases, so no issue.

For example, if x=y, and residue same, but we don't have any case where a=b, and in our satisfying pairs, a and b are different, so x and y cannot be equal in these cases.

But even if they were, but in this set, since residues are different, x≠y.

But not a problem.

Now, the answer should be 400.

But let me check the options: A)50, B)100, C)200, D)400, E)500. So D)400.

But the problem says "two-digit natural numbers less than 50", which we took as 10 to 49 inclusive.

Is 50 included? It says less than 50, and two-digit, so 50 is not included, and 50 is not two-digit? 50 is two-digit, but less than 50? 50 is not less than 50, so excluded.

And we have 10 to 49, which is correct.

Number of elements: 40, as calculated.

Residues: each has 10, which seems symmetric.

But is 50 included? No.

What about 10? Included.

But let me confirm the number with residue 0: 12,16,...,48.

12/4=3, 48/4=12, so multiples from 3 to 12 inclusive? 3,4,5,6,7,8,9,10,11,12? That's 10 values, yes.

Similarly for others.

Another way: the smallest number 10, largest 49.

10 div 4 rem 2.

11 rem 3.

12 rem 0.

13 rem 1.

14 rem 2.

...

Up to 49 rem 1? 48 rem 0, 49 rem 1.

Now, from 10 to 49, total numbers: 40.

The residues cycle every 4 numbers: 2,3,0,1, then repeat.

Sequence: 10:2,11:3,12:0,13:1,14:2,15:3,16:0,17:1, etc.

Each block of 4 consecutive numbers has residues 2,3,0,1.

All distinct and cover all residues.

Now, from 10 to 49 inclusive, number of such blocks.

First number 10, last 49.

The sequence starts at 10.

The residues: position: if we consider modulo 4, but perhaps count how many full cycles.

From 10 to 49, the numbers are consecutive.

The smallest is 10, largest 49.

Total numbers: 40.

Since 40 is divisible by 4, and the residues are periodic with period 4, and each residue appears equally often.

In each block of 4 consecutive integers, each residue appears once.

Here, 40 numbers, so exactly 10 per residue, yes.

So, no issue.

Now, so total pairs: 400.

But is there any constraint that x and y are distinct? The problem doesn't say, but in our pairs, since residues are different, x and y are different, but even if they could be same, but in this case they are not.

But for example, if we consider (x,y) where x and y same, but only if i^x + i^x = 2i^x =0, impossible, so no solution when x=y, and in our cases, since residues differ, x≠y automatically.

So, no problem.

Perhaps the problem considers unordered pairs, but that would be unlikely, because the equation is not symmetric, and in the conditions, we have both (0,2) and (2,0), which are different.

Moreover, if unordered, then we would have to halve it, but 400/2=200, which is option C, but that might be incorrect.

Let me see.

For example, take x=10, y=12: i^{10} + i^{12} = -1 + 1 =0.

Now, if we consider unordered pair {x,y}, but {10,12} is one pair.

But when we take x=12, y=10: i^{12} + i^{10} =1 + (-1)=0, same pair in unordered sense? But in ordered pair, (10,12) and (12,10) are different.

But in the equation, both satisfy: for (x,y)=(10,12), i^x + i^y = i^{10} + i^{12} = -1 + 1 =0.

For (12,10), i^{12} + i^{10} =1 + (-1)=0, same.

So both are solutions.

If we consider unordered pairs, we would count only once, but in the problem, it says "pairs of (x,y)", which typically means ordered pairs, especially since it's a pair in the context of the equation, and x and y are distinct variables.

Moreover, in the answer choices, 400 is there, and it makes sense.

If it were unordered, it would be half, but then 200, but let's see if that makes sense.

In our cases, for each pair of distinct residues that satisfy, we have multiple pairs.

But notice that the pairs (a,b) that satisfy are symmetric in the sense that if (a,b) works, so does (b,a), but not for the same x,y.

For example, for residue pair (0,2), we have pairs where x≡0, y≡2.

For (2,0), x≡2, y≡0.

These are disjoint sets of (x,y) pairs.

For each such pair of residue classes, since the classes have same size, we have 10*10=100 for each direction.

In the unordered pair of sets, but for the pairs (x,y), it's ordered.

Another way: total number of ways.

For a fixed x, what y satisfy the equation.

Given x, with residue a, then i^x is fixed, and we need i^y = -i^x.

Depending on a.

If a=0, i^x=1, so i^y=-1, so y≡2 mod 4.

Similarly, if a=1, i^x=i, i^y=-i, y≡3.

If a=2, i^x=-1, i^y=1, y≡0.

If a=3, i^x=-i, i^y=i, y≡1.

Now, for each x, there is exactly one residue class that y must be in.

Specifically:

If x≡0, y≡2

x≡1, y≡3

x≡2, y≡0

x≡3, y≡1

Now, and for each such y, since the residue class has 10 elements, and y is in the set.

So, for each x, there are exactly 10 y's that satisfy, regardless of what x is.

Is that true?

Number of y with the required residue.

Each residue class has 10 elements, so yes, for each x, there are 10 y's that work.

Now, total x: 40.

So total pairs: 40 * 10 = 400.

Same answer.

Since for each x, exactly 10 choices for y.

Now, is there any x for which no y? But no, as above.

And when we fix x, we get 10 y's, and since y is independent, no overcount.

So total $40 * 10 = 400$.

Now, if we think about y, same thing: for each y, how many x?

Similarly, if y≡0, then i^y=1, so i^x must be -1, so x≡2.

If y≡1, i^y=i, i^x must be -i, x≡3.

If y≡2, i^y=-1, i^x must be 1, x≡0.

If y≡3, i^y=-i, i^x must be i, x≡1.

So for each y, there is exactly one residue class for x, with 10 elements.

So again, 40 y's, each has 10 x's, total 400.

Same.

Now, but is there any pair where x=y? But as before, no, because if x=y, and say x≡0, then i^x=1, i^y=1, sum=2≠0, and the required y would be ≡2, different.

But in our count, for each x, we choose y different, but since the equation is satisfied only when y is in that class, and x is not in that class for that residue, but anyway, we are counting ordered pairs correctly.

Now, so 400 seems correct.

But why is there an option 200? Maybe if someone thinks unordered pairs, but that would be 200, but as we saw, each unordered pair {x,y} that satisfies might be counted twice in ordered pairs, but only if x≠y, which is always true in our case.

For each pair of distinct x,y that satisfy, since the equation is symmetric in a way? No.

Suppose {x,y} such that for the unordered pair, the equation holds.

But in our case, for a pair {x,y}, when does i^x + i^y =0?

This happens if the residues satisfy one of the conditions, but since the pair is unordered, we need to see.

For example, take x and y with residues such that |a-b|=2, as before.

For a given pair {x,y} with |a-b|=2, then both ordered pairs (x,y) and (y,x) satisfy the equation.

Is that true?

Suppose x≡0, y≡2.

Then (x,y): i^x + i^y =1 + (-1)=0.

(y,x): i^y + i^x = -1 + 1=0, same.

Similarly, if x≡1, y≡3: i^x=i, i^y=-i, sum i + (-i)=0.

(y,x): same.

So yes, for every such unordered pair {x,y} with residues differing by 2, both ordered pairs satisfy.

Moreover, if x=y, but we saw no solution, and residues same, so not possible.

Now, how many such unordered pairs?

Total number of ways to choose two distinct x and y with |a-b|=2.

The residue pairs that work are the pairs of residues: (0,2), (1,3), and also, but (0,2) and (2,0) are the same unordered residue pair, since residues are unordered in this context.

For unordered pair of residues, the pairs that satisfy |a-b|=2 are: {0,2}, {1,3}.

That's it, since {2,0} is same as {0,2}, etc.

And {0,2} and {1,3} are the only unordered pairs of distinct residues with difference 2.

Now, for each such residue pair, say {0,2}, how many unordered pairs {x,y} with one in res 0, one in res 2.

Since x and y are distinct, and different classes.

Number: there are 10 choices for res 0, 10 for res 2, but since unordered, each pair {x,y} is counted once, so 10*10=100 pairs for {0,2}.

Similarly for {1,3}: 10*10=100 pairs.

So total unordered pairs that satisfy: 100 + 100 = 200.

But each such unordered pair corresponds to two ordered pairs, and the equation holds for both ordered pairs.

But in the problem, it asks for "pairs of (x,y)", and typically in such contexts, especially since it's an equation with two variables, it probably means ordered pairs.

Moreover, if it were unordered, it should specify "unordered pairs" or "sets", but it says "pairs (x,y)", which suggests ordered.

Additionally, in the answer choices, 400 is there, and it makes sense.

Furthermore, if we think about the total number.

Another thing: in our calculation, for each x, there is a y that works, but multiple y's.

Now, the minimal and maximal.

x ranges from 10 to 49, y same.

Now, total satisfying pairs 400.

Is there any constraint that x and y are distinct? But as we saw, they are always distinct in the solution, but even if not, but in this case, no issue.

Perhaps the problem considers only distinct x and y, but the equation could hold with x=y, but it doesn't, so same.

Or perhaps they include single, but no.

Another thought: are there pairs where i^x + i^y =0 with same residue? But we saw no, since if same residue, i^x = i^y, so 2i^x=0, impossible.

So only when residues differ by 2.

Now, so I think 400 is correct.

But let me see the answer choices; perhaps I miscounted the set.

Two-digit natural numbers less than 50: 10 to 49 inclusive.

But 10 is included? Yes.

Number: 49 - 10 + 1 = 40, yes.

Each residue: 10, as calculated.

Perhaps they exclude 10? But 10 is two-digit.

Or include 50? But less than 50, so no.

Perhaps "less than 50" includes 50? But no, less than 50 means less than 50.

In some contexts, natural numbers might start from 0 or 1, but two-digit, so must be at least 10.

Another thing: is 10 considered? i^{10} = (i^2)^5 = (-1)^5 = -1, yes.

y=12, i^{12}=1, sum 0.

Now, total pairs: we have for each of the 40 x, 10 y's.

But for a fixed x, say x=10, res 2, then y must be res 0, so y=12,16,20,24,28,32,36,40,44,48. That's 10 y's.

Similarly for others.

So yes.

Total: 40 * 10 = 400.

If we list the required y for each x.

Since the mapping is consistent.

But I think it's correct.

Perhaps the problem is that when we say "pairs (x,y)", and if x and y are symmetric, but in this equation, it is symmetric, since i^x + i^y = i^y + i^x, so the equation is symmetric in x and y, meaning that if (x,y) satisfies, so does (y,x).

Therefore, the solution set is symmetric, so if we count ordered pairs, we get pairs where x and y are distinct and satisfy.

But in this case, all solutions have $x \neq y$, as we saw.

And for each solution pair (x,y) with $x \neq y$, since the residues are different and $|a-b|=2$, and as above, (y,x) is also a solution, and it's different.

In fact, for each such pair, there is a symmetric one.

But in our count, we have both, total 400.

The number of distinct unordered pairs would be half of that, 200, since no fixed points.

But again, I think the problem asks for ordered pairs.

To confirm, suppose we have a smaller case.

Suppose the set is smaller, say two-digit numbers less than 14, so 10,11,12,13.

Then residues: 10≡2,11≡3,12≡0,13≡1.

Each residue once.

Now, which pairs (x,y) satisfy i^x + i^y=0.

Possible pairs:

List all ordered pairs.

x=10, i^x=-1, need i^y=1, so y must be ≡0, so y=12. So (10,12)

x=11, i^x=-i, need i^y=i, so y≡1, y=13. (11,13)

x=12, i^x=1, need i^y=-1, so y≡2, y=10. (12,10)

x=13, i^x=i, need i^y=-i, so y≡3, y=11. (13,11)

Now, so the satisfying pairs: (10,12), (11,13), (12,10), (13,11). So four pairs.

Total possible ordered pairs: $4 * 4 = 16$.

Now, according to our calculation: for each x, one y, total 4*1? But in this case, each residue has one element, but in our earlier, for each x, number of y is the size of the required residue class, which here is 1 for each.

So $4 * 1 = 4$? But we have four pairs, but for each x, there is one y, so total 4 pairs, which matches what we listed.

Now, unordered pairs: {10,12}, {11,13}. Two unordered pairs.

But in the list, we have two pairs, but each corresponds to two ordered pairs.

Now, in the problem, if it asked for unordered pairs, it would be 2, but here we have 4 ordered pairs.

Now, the answer should be 4.

But in the actual problem, with 40 elements, if we compute, with 4 elements, we have 4 ordered pairs satisfying.

In our formula, since each residue class size is 1, and we have four cases: for each of the four residue pairs, but wait.

In general, with n elements per residue, but here n=1.

Total satisfying pairs: since we have the four combinations: (res x, res y) = (0,2),(2,0),(1,3),(3,1), each with one x and one y, so each gives one pair, total 4.

Since 4 x's, each has one y, total 4.

Now in the actual problem, with 10 per residue, so 40 x's, each has 10 y's, total 400.

If the problem intended unordered pairs, then in this small case, it should be 2, but the problem likely expects ordered pairs.

Moreover, in the answer choices, for the actual problem, if it were unordered, 200 is there, but 200 is less than 400, and in our small case, 2 vs 4, so unordered would be half.

But in the actual problem, with 40 elements, total unordered pairs are C(40,2) for distinct pairs, but including pairs where x=y not, but since no solution when equal, but in our satisfying, we have only some pairs.

Total satisfying unordered pairs: as above, for each residue pair {0,2} and {1,3}, number of ways to choose one from each set.

For {0,2}, 10 choices for set 0, 10 for set 2, but since unordered, and x and y distinct, each pair {x,y} with x in A, y in B, and since A and B disjoint, the number is |A|*/B/ =10*10=100 for {0,2}.

Similarly 100 for {1,3}, total 200 unordered pairs.

Each such unordered pair satisfies the equation, because if x≡0, y≡2, then i^x + i^y=1 + (-1)=0.

Similarly for {1,3}.

And no others.

So yes, 200 unordered pairs satisfy.

But the problem says "pairs (x,y)", and in mathematics, when we say pairs, it can be ambiguous, but often in such contexts, especially with equations, ordered pairs are considered.

Moreover, in the answer choices, both 200 and 400 are present, so likely they expect ordered pairs.

To confirm, let's see the problem statement: "how many different pairs of (x,y) are there"

It explicitly writes (x,y) in parentheses, which often denotes ordered pair.

Additionally, in combinatorics problems, when they say "pairs", it might be unordered, but here they specify (x,y), so probably ordered.

In our small example with 4 elements, if we have 4 ordered pairs satisfying, and if the answer should be 4, which is larger, while if unordered it would be 2.

Now, in the actual problem, with 40 elements, total ordered pairs are 1600, satisfying 400, which is 1/4.

If we had included x=y, but there are no solutions when x=y.

Another way: total number of solutions.

Since the condition is that x and y have residues differing by 2 mod 4.

As we have.

The probability is 1/4, as before.

I recall that in some similar problems, they consider ordered pairs.

Moreover, looking at the answer choices, 400 seems plausible.

Perhaps I miscounted the number of elements.

Two-digit natural numbers less than 50: from 10 to 49 inclusive.

But 10 to 49: first number 10, last 49, number is 49 - 10 + 1 = 40, standard.

Inclusive, so yes.

Residues: as computed, each has 10.

But is that exact?

The smallest number: 10.

10 div 4: 4*2=8, remainder 2.

11: 4*2=8, rem 3.

12: rem 0.

13: rem 1.

...

48: rem 0 (since 48/4=12)

49: 48+1, rem 1.

Now, from 10 to 49, inclusive.

The sequence: 10,11,12,13,14,...,48,49.

Total numbers: 40.

The residues: 2,3,0,1,2,3,0,1,... repeating every 4.

Since 40 divisible by 4, and each block of 4 has one of each residue, so exactly 40/4=10 per residue.

Yes.

If it were up to 50, but less than 50, so 50 not included.

50 would be ≡2 mod 4, since 50/4=12*4=48, rem 2.

But not included.

So, I think it's correct.

Perhaps "natural numbers" include 0? But two-digit, so no, since 00 is not two-digit, usually.

So, 10 to 49.

Another thought: is 10 included? Yes.

But in some definitions, natural numbers start from 0 or 1, but two-digit, so 10 to 99, but here less than 50, so 10 to 49.

Yes.

So, I think 400 is correct.

But let me see why one might think 200.

If someone assumes that the pair is unordered, or if they think that for each pair of residue classes, they count the number of unordered pairs.

But in that case, for each residue pair like (0 and 2), they have 10*10=100 unordered pairs, but since {0,2} and {1,3} are two such pairs, total 200.

But as we saw, that would be for unordered pairs.

But in the problem, it specifies (x,y), so likely not.

Perhaps they consider x and y without regard to order, but the problem says "pairs (x,y)", which is ordered.

Another mistake: perhaps they think that i^x + i^y=0 implies that the sum is zero, and perhaps they miss some cases.

For example, if they only consider when i^x =1 and i^y=-1, but not the others.

But we have four cases.

In our count, we have 100 for each of the four subcases.

But all are valid.

Perhaps they exclude when x or y is multiple of something, but no.

Another thing: the exponent could be 0, but here x and y are at least 10, so no issue.

i^0=1, but no zero exponent here.

So, I think it's fine.

Perhaps "less than 50" is misinterpreted, but I think it's correct.

Or two-digit includes 50? But less than 50, so no.

Perhaps up to 49, but 49 is included, yes.

Now, so I think the answer is 400.

So, option D.

But let me see the answer choices: A)50, B)100, C)200, D)400, E)500.

500 is also there, which is more than 400.

How could it be 500? If they included more numbers.

For example, if they included numbers from 10 to 50, but 50 is not less than 50.

Or if they started from 00 or something, but no.

Total numbers: 40, if they thought 50 is included, then from 10 to 50 inclusive: 10 to 50 is 41 numbers? 50-10+1=41.

Then residues: now 41 numbers.

10 to 50.

50≡2 mod 4, as above.

So residues: from 10 to 49: 40 numbers, 10 each residue.

Add 50≡2, so res 2 now has 11 elements, others 10.

Then, for the pairs.

For example, for case a=0,b=2: x res 0:10, y res 2:11, so 10*11=110.

Similarly, a=2,b=0: x res 2:11, y res 0:10, so 11*10=110.

a=1,b=3: x res 1:10, y res 3:10, 100.

a=3,b=1: x res 3:10, y res 1:10, 100.

Total: 110+110+100+100=420, not 500.

If they included single digit, but two-digit, no.

Or if natural numbers include 1 to 49, but two-digit, so 10 to 49.

Perhaps they consider x and y can be same, but still not.

> Another mistake: if they think that for each x, the required residue is fixed, and they multiply by the number, but if they miscount the residue sizes.
>
> But in this case, symmetric.
>
> Perhaps they think that the period is 2, but no.
>
> Or confuse with real numbers, but no.
>
> I think 400 is correct.
>
> So, I'll go with that.

The equation $i^x + i^y = 0$ implies $i^x = -i^y$. Since the powers of the imaginary unit i cycle every 4 exponents, the behavior depends on the exponents modulo 4. The values are:

$i^k = 1$ if $k \equiv 0 \pmod 4$

$i^k = i$ if $k \equiv 1 \pmod 4$

$i^k = -1$ if $k \equiv 2 \pmod 4$

$i^k = -i$ if $k \equiv 3 \pmod 4$

The equation $i^x + i^y = 0$ holds in the following cases for residues modulo 4:

$i^x = 1$ and $i^y = -1 \to x \equiv 0 \pmod 4$, $y \equiv 2 \pmod 4$

$i^x = -1$ and $i^y = 1 \to x \equiv 2 \pmod 4$, $y \equiv 0 \pmod 4$

$i^x = i$ and $i^y = -i \to x \equiv 1 \pmod 4$, $y \equiv 3 \pmod 4$

$i^x = -i$ and $i^y = i \to x \equiv 3 \pmod 4$, $y \equiv 1 \pmod 4$

Equivalently, the residues $a = x \mod 4$ and $b = y \mod 4$ must satisfy $|a - b| = 2$, which corresponds to the pairs (a, b): $(0, 2)$, $(2, 0)$, $(1, 3)$, $(3, 1)$.

The values of x and y are two-digit natural numbers less than 50, so they range from 10 to 49 inclusive. The number of such numbers is $49 - 10 + 1 = 40$.

Residue 0: 12, 16, ..., 48 → 10 numbers

Residue 1: 13, 17, ..., 49 → 10 numbers

Residue 2: 10, 14, ..., 46 → 10 numbers

Residue 3: 11, 15, ..., 47 → 10 numbers

The pairs (x, y) are ordered, and the solution pairs correspond to the residue pairs above. The number of pairs for each case is:

Case 1 ($x \equiv 0$, $y \equiv 2$): $10 \times 10 = 100$

Case 2 ($x \equiv 2$, $y \equiv 0$): $10 \times 10 = 100$

Case 3 ($x \equiv 1$, $y \equiv 3$): $10 \times 10 = 100$

Case 4 ($x \equiv 3$, $y \equiv 1$): $10 \times 10 = 100$

Summing these gives the total number of ordered pairs: $100 \times 4 = 400$.

Alternatively, for each fixed x, there is exactly one residue class for y that satisfies the equation, and since each residue class has 10 elements, there are 10 choices for y for each x. With 40 values for x, the total is $40 \times 10 = 400$.

The equation is symmetric in x and y, and all solutions satisfy $x \neq y$, but the problem specifies ordered pairs (x, y), and the count is consistent.

The answer is 400, corresponding to option D.

\boxed{400}

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