There is an ABC triangle and a point D on the edge [BC] between the vertices B and C. If the lengths of the edges |AB|, |BD|, and |AC| are equal to one another, |AB| = |BD| = |AC|, what is the maximum integer value of the angle ∡BAD?

Answer from Mathful

Let's analyze the problem step-by-step:

Given:

Triangle ABC

Point D lies on segment $[BC]$ between B and C.

Lengths satisfy:

$$|AB| = |BD| = |AC|$$

We want the maximum integer value of the angle $\angle BAD$.

Step 1: Understand the setup

We have:

$AB = AC$ means triangle ABC is isosceles with $AB = AC$.

$BD = AB = AC$.

D lies on BC, between B and C.

So, $BD = AB = AC = \ell$ (say).

Step 2: Geometric configuration

Since $AB = AC$, triangle ABC is isosceles with apex A.

D is on BC, such that $BD = AB$.

We want to find the angle at A between AB and AD.

Step 3: Assign coordinates

To analyze the problem, assign coordinates to points:

Place B at the origin: $B = (0, 0)$.

Place C on the x-axis: $C = (c, 0)$, for some $c > 0$.

Since D is on BC, $D = (d, 0)$ with $0 < d < c$.

Step 4: Point A coordinates

Since $AB = AC = \ell$, A lies on the perpendicular bisector of BC.

Midpoint of BC is $M = \left(\frac{c}{2}, 0\right)$.

$A = \left(\frac{c}{2}, h\right)$, $h > 0$.

Step 5: Express the lengths

$AB = \ell$:

$$AB^2 = (0 - \frac{c}{2})^2 + (0 - h)^2 = \left(\frac{c}{2}\right)^2 + h^2 = \ell^2$$

$AC = \ell$:

$$AC^2 = (c - \frac{c}{2})^2 + (0 - h)^2 = \left(\frac{c}{2}\right)^2 + h^2 = \ell^2$$

This confirms A lies on the perpendicular bisector.

$BD = \ell$:

$$BD = d - 0 = d = \ell$$

So,

$$d = \ell$$

Step 6: Find $\angle BAD$

Vectors at A:

Vector $\overrightarrow{AB} = B - A = (0 - \frac{c}{2}, 0 - h) = \left(-\frac{c}{2}, -h\right)$

Vector $\overrightarrow{AD} = D - A = (d - \frac{c}{2}, 0 - h) = \left(d - \frac{c}{2}, -h\right)$

Recall:

$d = \ell$

$\ell^2 = \left(\frac{c}{2}\right)^2 + h^2$

Step 7: Calculate $\cos\angle BAD$

$$\cos\angle BAD = \frac{\overrightarrow{AB} \cdot \overrightarrow{AD}}{|\overrightarrow{AB}| \cdot |\overrightarrow{AD}|}$$

Calculate numerator:

$$\overrightarrow{AB} \cdot \overrightarrow{AD} = \left(-\frac{c}{2}\right)\left(d - \frac{c}{2}\right) + (-h)(-h) = -\frac{c}{2}\left(d - \frac{c}{2}\right) + h^2$$

Calculate magnitudes:

$|\overrightarrow{AB}| = \sqrt{\left(-\frac{c}{2}\right)^2 + (-h)^2} = \sqrt{\left(\frac{c}{2}\right)^2 + h^2} = \ell$

$|\overrightarrow{AD}| = \sqrt{\left(d - \frac{c}{2}\right)^2 + h^2}$

Step 8: Express in terms of c and h

Recall $d = \ell = \sqrt{\left(\frac{c}{2}\right)^2 + h^2}$.

Substitute:

$$\overrightarrow{AB} \cdot \overrightarrow{AD} = -\frac{c}{2}\left(\sqrt{\left(\frac{c}{2}\right)^2 + h^2} - \frac{c}{2}\right) + h^2$$

Step 9: Simplify numerator

Let $x = \frac{c}{2}$, so $\ell = \sqrt{x^2 + h^2}$.

Numerator:

$$N = -x(\ell - x) + h^2 = -x\ell + x^2 + h^2$$

But $\ell^2 = x^2 + h^2$, so:

$$N = -x\ell + \ell^2 = \ell^2 - x\ell = \ell(\ell - x)$$

Step 10: Denominator

$$|\overrightarrow{AB}| = \ell$$

$$|\overrightarrow{AD}| = \sqrt{(d - x)^2 + h^2} = \sqrt{(\ell - x)^2 + h^2}$$

But recall $h^2 = \ell^2 - x^2$, so:

$$|\overrightarrow{AD}| = \sqrt{(\ell - x)^2 + \ell^2 - x^2} = \sqrt{\ell^2 - 2\ell x + x^2 + \ell^2 - x^2} = \sqrt{2\ell^2 - 2\ell x}$$

$$|\overrightarrow{AD}| = \sqrt{2\ell(\ell - x)}$$

Step 11: Compute cosine

$$\cos \angle BAD = \frac{N}{|\overrightarrow{AB}||\overrightarrow{AD}|} = \frac{\ell(\ell - x)}{\ell\sqrt{2\ell(\ell - x)}} = \frac{\ell - x}{\sqrt{2\ell(\ell - x)}}$$

Simplify:

$$\cos \angle BAD = \frac{\ell - x}{\sqrt{2\ell(\ell - x)}} = \frac{\sqrt{\ell - x}}{\sqrt{2\ell}} = \sqrt{\frac{\ell - x}{2\ell}}$$

Step 12: Introduce parameter $t = \frac{x}{\ell}$

Since $x = \frac{c}{2}$, and $\ell = \sqrt{x^2 + h^2}$, we have:

$$t = \frac{x}{\ell} \in (0, 1)$$

Then

$$\cos \angle BAD = \sqrt{\frac{\ell - x}{2\ell}} = \sqrt{\frac{1 - t}{2}}$$

Step 13: Maximize $\angle BAD$

$$\cos \angle BAD = \sqrt{\frac{1 - t}{2}}, \quad t \in (0, 1)$$

When $t \to 0$,

$$\cos \angle BAD \to \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} \approx 0.707$$

so

$$\angle BAD \to 45^\circ$$

When $t \to 1$,

$$\cos \angle BAD \to 0$$

so

$$\angle BAD \to 90^\circ$$

Hence, the angle $\angle BAD$ can range from 45° to 90°.

Step 14: Interpretation of $t \to 1$

$t = \frac{x}{\ell} = \frac{c/2}{\ell} \to 1$ means $c/2 \to \ell$ or $c \to 2\ell$.

At this limit, $h \to 0$, so A moves closer to the segment BC.

Conclusion:

Maximum $\angle BAD$ approaches 90° but never reaches it exactly because A must be above the segment BC (triangle ABC must have positive area).

The maximum integer value less than or equal to 90° is 89 degrees.

Final answer:

$$\boxed{89}$$

The maximum integer value of the angle $\angle BAD$ is 89 degrees.

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